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Spin and angular momentum in gravity waves

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The angular momentum A per unit horizontal distance of a train of periodic, progressive surface waves is a well-defined quantity, independent of the horizontal position of the origin of moment.

The Lagrangian-mean angular momentum $\bar{A}_L$ consists of two parts, arising from the orbital motion and from the Stokes drift respectively. Together these contribute a positive sum, nearly proportional to the energy density (when the origin is taken in the mean surface level). If moments are taken about some point P not at the mean surface level, the angular momentum will differ by an amount proportional to the elevation of P. There is just one elevation for which the Lagrangian-mean angular momentum about P vanishes. This elevation is called the level of action. For infinitesimal waves in deep water the level of action is at a height above the mean surface equal to $1/2k$, that is $1/4\pi$ times the wavelength.

Just as for ordinary fluid velocities, the Lagrangian-mean angular momentum $\bar{A}_L$ differs from the Eulerian-mean $\bar{A}_L$, the latter being zero to second order. The difference between $\bar{A}_L$ and $\bar{A}_E$ is associated with the displacement of the lateral boundaries of any given mass of fluid.

For waves of finite amplitude, an initially rectangular mass of fluid becomes ultimately quite distorted by the Stokes drift. Nevertheless it is possible to define a long-time average l.t.$\bar{A}$ and to calculate its numerical value accurately in waves of finite amplitude. In low waves, l.t.$\bar{A}$ is equal to $\bar{A}_L$. Defining the level of action y_a in the general case as l.t.$\bar{A}/I$, where I is the linear momentum, we find that y_a rises from $0{\cdot}5k^{-1}$ for infinitesimal waves to about $0{\cdot}6k^{-1}$ for steep waves. Thus y_a is about the same as the height $y_{\max}$ of the wave crests above the mean level in limiting waves, a fact which may account for why steep irrotational waves can support whitecaps in a quasi-steady state. The same argument suggests that Gerstner waves (in which the particle orbits are theoretically circular) could not support whitecaps.

1. Introduction

Certain integral properies of water waves, notably their momentum, energy, radiation stress and energy flux, are known to play an important role in the interpretation of surface wave phenomena, and have been studied recently as functions of the wave amplitude (Longuet-Higgins 1974, 1975; Longuet-Higgins & Fenton 1974; Cokelet 1977). In this paper we draw attention to another property, less well known, which nevertheless appears to be associated with equally interesting phenomena, namely the angular momentum of a wave train about any given point.

Because the particles in a deep-water wave describe roughly circular orbits one would intuitively expect the wave train to possess, on average, a positive angular momentum, when considered from a Lagrangian point of view. As shown in § 3, the orbital angular momentum of a wave train of speed c and amplitude a is indeed positive and equal to $\frac{1}{2}\rho a^2 c$, to second order. But the steady mean drift of the particles in an irrotational wave gives a negative contribution $-\frac{1}{4}\rho a^2 c$, so that altogether, for waves of small amplitude, there is a net positive angular momentum $\frac{1}{4}\rho a^2 c$ about any point in the mean level.

The angular momentum about a point at any other level y will differ by an amount proportional to y times the total horizontal momentum in the wave (the mean vertical momentum being zero). In particular there is one level about which the Lagrangian angular momentum vanishes. For waves of low amplitude this is at a height $L/4\pi$ above the mean surface level, where L is the wavelength. It is suggested that waves must attain this height before they can support a whitecap in a quasi-steady state. It turns out that in a steep, irrotational wave in deep water the wave crest does practically attain the necessary height (see figure 4 in § 9).

The evaluation of the angular momentum reveals certain paradoxes, similar to those encountered in other water-wave phenomena. For instance the Eulerian-mean angular momentum is not second order but fourth order and therefore much smaller (§ 6). The difference is similar to that encountered with a simpler entity, the time-averaged velocity. Thus the Lagrangian-mean velocity in an irrotational wave is positive, whereas the Eulerian-mean velocity is precisely zero. The resolution of the paradox in the case of the mean angular momentum is discussed in § 6.

We have quoted the value $\frac{1}{4}\rho a^2 c$ for the Lagrangian-mean angular momentum in waves of small amplitude. For waves of finite amplitude it is found possible to define a long-time average of the Lagrangian angular momentum (see § 8) and to calculate it precisely (§ 9). Numerical values are given in table 2.

It is worth comment that the *dimensions* of angular momentum are the same as those of wave *action* (see § 10). Thus although the two entities are quite distinct, the known conservation of wave action in weakly nonlinear wave interactions implies also the conservation of (Lagrangian) angular momentum.

2. Angular momentum: general considerations

Consider a periodic, irrotational gravity wave of wavelength L in water of uniform mean depth h, progressing with speed c as in figure 1. Let (x, y) be rectangular co-ordinates with the origin in the mean level, the x axis in the direction of wave propagation and the y axis vertically upwards. Denoting by (u, v) the horizontal and vertical components of the velocity, we may choose a frame of reference in which the mean value of u is zero at some given point below the wave trough. Thus we have

$$\overline{u} = 0, \tag{2.1}$$

where a bar denotes the average over one wavelength. If ϕ is the velocity potential $((u, v) = \nabla\phi)$ then it follows that

$$[\phi]_{x=0}^{L} = 0 \tag{2.2}$$

so that if (2.1) is true at one level y it is true at all other such levels.

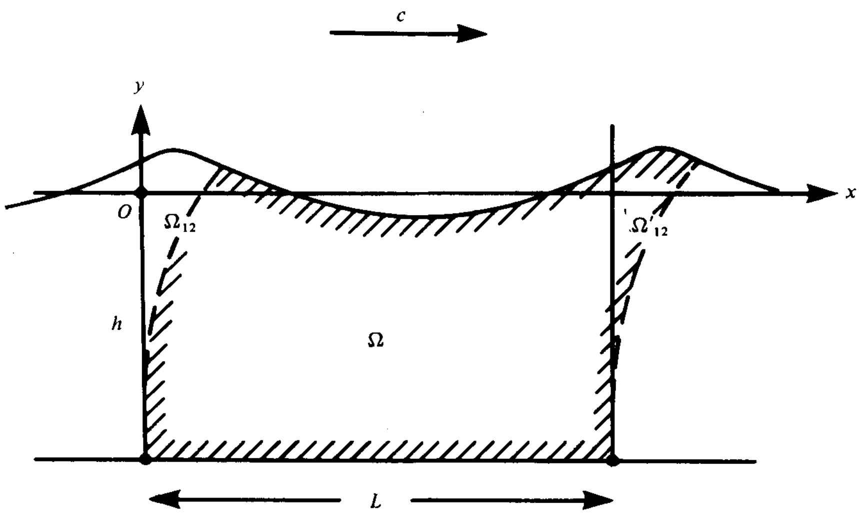

FIGURE 1. Definition sketch, showing co-ordinates and boundaries of the fluid.

Now consider the angular momentum of the fluid contained in the space Ω between the free surface $y = y_s(x, t)$, the bottom $y = -h$ and two curves $x = f(y)$ and $x = f(y) + L$ separated horizontally by a distance L. Let $LA(x_0)$ denote the angular momentum about an arbitrary point $(x_0, 0)$ in the mean level:

$$LA(x_0) = \iint_\Omega [yu - (x - x_0)\, v]\, dx\, dy. \tag{2.3}$$

Clearly

$$LA(x_0) = LA(0) + x_0 \iint_\Omega v\, dx dy. \tag{2.4}$$

The last integral represents the total vertical momentum over one wavelength, which must vanish identically. Hence A is independent of x_0 and we may write simply

$$LA = \iint_\Omega (yu - xv)\, dx dy \tag{2.5}$$

for the angular momentum about any point in the mean surface level. Clearly A is the mean angular momentum per unit horizontal distance.

However, A may depend upon f. Thus if $x = f_1(y)$ and $x = f_2(y)$ are two distinct bounding curves, then the difference between the corresponding angular momenta LA_1 and LA_2 is

$$L(A_2 - A_1) = \iint_{\Omega'_{12}} (yu - xv)\, dx\, dy - \iint_{\Omega_{12}} (yu - xv)\, dx\, dy, \tag{2.6}$$

where Ω_{12} is the region bounded by the curves $x = f_1(y)$ and $x = f_2(y)$, and Ω'_{12} is the region bounded by $x = f_1(y) + L$ and $x = f_2(y) + L$. But by the space-periodicity the motion in Ω'_{12} is exactly similar to the motion in Ω_{12}. So we can transform Ω'_{12} into Ω_{12} on replacing x by $(x - L)$ and (2.6) becomes

$$L(A_2 - A_1) = \iint_{\Omega_{12}} (-Lv)\, dx\, dy \tag{2.7}$$

or

$$\Delta A = A_2 - A_1 = \iint_{\Omega_{12}} (-v)\, dx\, dy. \tag{2.8}$$

The two values of the average angular momentum thus differ by precisely the total *vertical* momentum contained in the zone between $y = y_s$, $y = -h$ and the two curves $x = f_1(y)$ and $x = f_2(y)$.

3. The Lagrangian-mean angular momentum

The co-ordinates (x, y) of a particle with fixed Lagrangian co-ordinates (α, β) are given as functions of α, β and the time t by

$$\left.\begin{aligned} x &= \alpha - a\,\frac{\cosh k(\beta+h)}{\sinh kh}\sin(k\alpha-\sigma t) + a^2k\,\frac{\cosh 2k(\beta+h)}{2\sinh^2 kh}\,\sigma t,\\ y &= \beta + a\,\frac{\sinh k(\beta+h)}{\sinh kh}\cos(k\alpha-\sigma t). \end{aligned}\right\} \tag{3.1}$$

In (3.1) the terms of order a represent the horizontal and vertical displacements of the particle from its mean position, while the term in a^2k represents the horizontal displacement due to the Stokes drift. Terms of higher order are neglected, as for example the Doppler-shift in frequency due to the mean drift, which is a function of β. To the same order, the bottom is given by $\beta = -h$ and the free surface by $\beta = 0$ (more accurately $\beta = \frac{1}{2}a^2k$).

Corresponding to (3.1) we have the particle velocities

$$\left.\begin{aligned} u &= a\sigma\,\frac{\cosh k(\beta+h)}{\sinh kh}\cos(k\alpha-\sigma t) + a^2k\sigma\,\frac{\cosh 2k(\beta+h)}{2\sinh^2 kh},\\ v &= a\sigma\,\frac{\sinh k(\beta+h)}{\sinh kh}\sin(k\alpha-\sigma t). \end{aligned}\right\} \tag{3.2}$$

We may substitute now into (2.5) and carry out the integration, noting that

$$\frac{\partial(x, y)}{\partial(\alpha, \beta)} = 1 + O(ak)^2 \tag{3.3}$$

so that to second order $dx\,dy$ may be replaced by $d\alpha\,d\beta$. Hence

$$\begin{aligned} LA_L &= \int_0^L\int_{-h}^0 (yu - xv)\, d\alpha\, d\beta \\ &= \int_0^L\int_{-h}^0 \left\{a^2\sigma\,\frac{\sinh 2k(\beta+h)}{\sinh^2 kh} + a^2\,k\sigma\beta\,\frac{\cosh 2k(\beta+h)}{2\,\sinh^2 kh}\right. \\ &\qquad\qquad \left. - a\sigma\alpha\,\frac{\sinh k(\beta+h)}{\sinh kh}\sin(k\alpha-\sigma t)\right\} d\alpha\, d\beta \\ &= L\left(\tfrac{1}{2}a^2c - \tfrac{1}{4}a^2c + \frac{ac}{k}\cos\sigma t\right), \end{aligned} \tag{3.4}$$

where $c = \sigma/k$, the phase speed. The first two terms in the bracket arise respectively from the elliptical motion of the particles and from the mean drift velocity or mass transport. Altogether then we have for the Lagrangian angular momentum

$$A_L = \tfrac{1}{4}a^2c + \frac{ac}{k}\cos\sigma t \tag{3.5}$$

and on average

$$\bar{A}_L = \tfrac{1}{4}a^2c \tag{3.6}$$

correct to second order.

An alternative but interesting method avoids the specific evaluation of integrals in (3.4). Taking mean values in (2.5) we have

$$\bar{A}_L = \frac{1}{L}\iint_\Omega \overline{(yu - xv)}\, dx\, dy. \tag{3.7}$$

Working to order a^2 as before we can set $x = \alpha + \int u_1\, dt$, $y = \beta + \int v_1\, dt$, $u = u_1 + u_s$, where (u_1, v_1) denote the first-order particle velocities and u_s is the mean Stokes drift velocity. It was shown in Longuet-Higgins (1953*b*) that

$$u_s = \frac{\partial \Psi}{\partial y}, \tag{3.8}$$

where Ψ is the stream function for the mass-transport velocity, namely

$$\Psi = \overline{u_1 \int v_1\, dt}. \tag{3.9}$$

So, dropping the suffixes, we have from (3.7)

$$\bar{A}_L = \int_{-h}^{0}\left(\overline{u\int v\, dt} - \overline{v\int u\, dt} + y\frac{\partial \Psi}{\partial y}\right) dy. \tag{3.10}$$

Using the general property that if P and Q are any two periodic quantities then

$$\overline{P\int Q\, dt} + \overline{Q\int P\, dt} = 0, \tag{3.11}$$

we have

$$\bar{A}_L = \int_{-h}^{0} 2\overline{u\int v\, dt}\, dy + [y\Psi]_{-h}^{0} - \int_{-h}^{0}\Psi\, dy. \tag{3.12}$$

From (3.9) it is clear that $\Psi = 0$ when $y = -h$. Hence

$$\bar{A}_L = \int_{-h}^{0} 2\Psi\, dy - \int_{-h}^{0}\Psi\, dy = \int_{-h}^{0}\Psi\, dy. \tag{3.13}$$

Now we can write

$$\Psi = -\overline{\frac{\partial \psi}{\partial y}\int \frac{\partial \psi}{\partial x}\, dt}, \tag{3.14}$$

where ψ is the first-order stream function. The motion being progressive we have $\partial/\partial x = -(1/c)\,\partial/\partial t$, and therefore

$$\Psi = \frac{1}{c}\overline{\psi\frac{\partial \psi}{\partial y}} = \frac{1}{2c}\frac{\partial}{\partial y}\overline{\psi^2}. \tag{3.15}$$

From (3.13) we have then

$$\bar{A}_L = \frac{1}{2c}[\overline{\psi^2}]_{y=-h}^{0} \tag{3.16}$$

and using the well-known expression

$$\psi = ac\frac{\sinh k(y+h)}{\sinh kh}\cos(kx-\sigma t) \tag{3.17}$$

we find immediately

$$\bar{A}_L = \tfrac{1}{4}a^2c \tag{3.18}$$

as before.

From (3.13) it is already obvious that the orbital motion contributes two times, and the drift velocity contributes minus one times, the final answer.

4. Rates of change: the dynamical equations

We can also account for the first-order time-dependent term in (3.5). Starting from the equations of motion in the form

$$\left.\begin{aligned}\frac{Du}{Dt} &= -\frac{\partial P}{\partial x},\\ \frac{Dv}{Dt} &= -\frac{\partial P}{\partial y},\end{aligned}\right\} \tag{4.1}$$

where D/Dt denotes differentiation following the motion

$$\frac{D}{Dt} = \frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}, \tag{4.2}$$

and where

$$P = p+gy, \quad (\rho = 1), \tag{4.3}$$

let us cross-multiply equations (4.1) by x and y. This yields

$$\frac{D}{Dt}(yu-vx) = -\left(y\frac{\partial P}{\partial x}-x\frac{\partial P}{\partial y}\right). \tag{4.4}$$

Now let each side be integrated over the area $\Omega(t)$ occupied by a moving mass of fluid with boundary $B(t)$. This gives

$$\frac{D}{Dt}\iint_\Omega (yu-xv)\,dx\,dy = -\iint_\Omega \left(y\frac{\partial P}{\partial x}-x\frac{\partial P}{\partial y}\right)dx\,dy. \tag{4.5}$$

By writing

$$y\frac{\partial P}{\partial x}-x\frac{\partial P}{\partial y} = \frac{\partial F}{\partial y}-\frac{\partial G}{\partial x}, \tag{4.6}$$

where

$$\left.\begin{aligned}F &= \tfrac{1}{2}(x^2+y^2)\frac{\partial P}{\partial x},\\ G &= \tfrac{1}{2}(x^2+y^2)\frac{\partial P}{\partial y},\end{aligned}\right\} \tag{4.7}$$

we may transform the right-hand side of equation (4.5) by Green's theorem into

$$\int_B (F\,dx+G\,dy), \tag{4.8}$$

the integral being taken clockwise round the contour B. Hence (4.5) becomes

$$\frac{D}{Dt}\iint_\Omega (yu-xv)\,dx\,dy = \int_B \tfrac{1}{2}(x^2+y^2)\,dP, \tag{4.9}$$

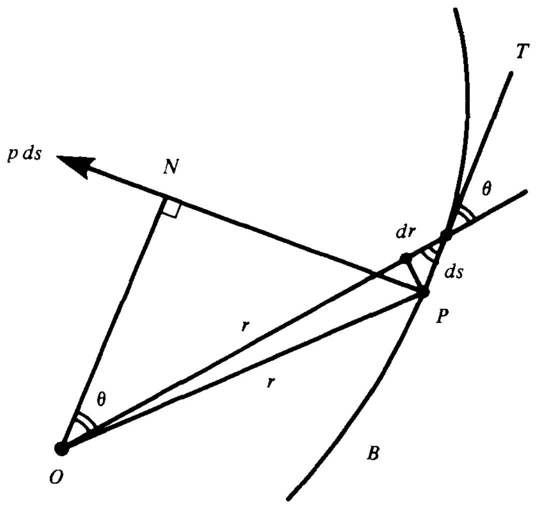

FIGURE 2. Sketch for the physical interpretation of the pressure integral in (4.11).

where P is given by (4.3). Integration by parts now gives

$$\frac{D}{Dt}\iint_{\Omega}(yu-xv)\,dx\,dy = -\int_B Pr\,dr, \tag{4.10}$$

where r denotes $(x^2+y^2)^{\frac{1}{2}}$.

Consider the interpretation of equation (4.10). If θ denotes the angle between the radius vector OP and the tangent PT to the boundary of B, as in figure 2, then we have $dr = ds\cos\theta$. Therefore, if N is the foot of the perpendicular from O to the normal PN we have

$$\int_B pr\,dr = \int_B pr\cos\theta\,ds = \iint_B ON\,p\,ds. \tag{4.11}$$

Hence the first part of the right-hand side of (4.10) represents the total moment about the origin of the pressure forces applied at the boundary.

The second part of the integral on the right-hand side of (4.10) can be written

$$-\int_B gy(x\,dx+y\,dy) = -\int_B gxy\,dx = \iint_{\Omega} gx\,dx\,dy, \tag{4.12}$$

that is $mg\bar{x}$, where m is the total mass (or volume) contained in Ω and $\bar{x}$ is the horizontal co-ordinate of its centre of mass. This then is the moment about O of the total gravitational force when applied at the centre of mass of the fluid. Equation (4.10) tells us that this is also equivalent to a normal pressure gy applied at the boundary.

Let us apply (4.10) to progressive, periodic waves in deep water. The integral on the right-hand side may be written

$$-\int_B (p+gy)\,(x\,dx+y\,dy). \tag{4.13}$$

In the part of the integral involving dy let us apply Bernoulli's theorem for steady irrotational flows:

$$p+gy = \tfrac{1}{2}(c^2-q^2), \tag{4.14}$$

where q denotes the local particle-speed in a reference frame moving with the phase-speed c. Substituting in (4.13) we see that the contribution from the term $\frac{1}{2}c^2$ vanishes. The term

$$\int_B \tfrac{1}{2}q^2 y\,dy \tag{4.15}$$

has four parts. The contribution from the horizontal bottom section ($dy = 0$) vanishes. The contributions from the two sides cancel each other, and at the free surface since $p = 0$ we have $\frac{1}{2}q^2 = g(y_0 - y)$, where y_0 is a constant. So this part of the integral gives

$$\int_{x=x_1}^{x_1+L} g(y_0 - y)\,y\,dy = g[\tfrac{1}{2}y_0 y^2 - \tfrac{1}{3}y^3]_{x=x_1}^{x_0+L} = 0 \tag{4.16}$$

by the periodicity. So (4.13) reduces to

$$-\int_B (p+gy)\,x\,dx. \tag{4.17}$$

The bottom section of the contour yields zero.† The two side sections together yield

$$-\int_f (p+gy)\,L\,dx = L\int_f \left\{\frac{\partial\phi}{\partial t} + \tfrac{1}{2}(u^2+v^2\right\} dx. \tag{4.18}$$

Finally, the free surface, where $p = 0$, yields

$$\int_{x_1}^{x_1+L} gy_s x\,dx = \int_0^L gy_s x\,dx - L\int_0^{x_1} gy_s\,dx. \tag{4.19}$$

Collecting together all the non-vanishing contributions to (4.13) we have from (4.10)

$$\frac{dA_L}{dt} = \int_0^{x_1}\left\{\frac{\partial\phi}{\partial t} + \tfrac{1}{2}(u^2+v^2)\right\} dx + \frac{1}{L}\int_0^L gy_s x\,dx - \int gy_s\,dx. \tag{4.20}$$

The terms on the right-hand side must together account for the time-dependence of the angular momentum (3.5).

To check this, note that in the first and third integrals in (4.20) the range of x is $O(a)$, hence the integrals are $O(a^2)$. In the second integral, however, we may set

$$y_s = a\cos(kx - \sigma t) \tag{4.21}$$

to order a, giving

$$\frac{dA_L}{dt} = -\frac{ag}{k}\sin\sigma t \tag{4.22}$$

in agreement with (3.5), since $\sigma c \doteqdot g$. Thus it appears that the fluctuation in angular momentum is due primarily to an integral along the free surface, representing the varying moment about the origin of the total gravitational force acting on the displaced fluid.

† This is because at infinite depth the pressure becomes hydrostatic, in a progressive wave (see Longuet-Higgins 1953*a*).

5. Application to breaking waves

We have so far considered the mean angular momentum density (a.m.d.) about an arbitrary point in the mean surface level $y = 0$. If we wish to calculate the a.m.d. about a point at some other level, say $y = y_0$, we have clearly

$$LA(y_0) = \iint_\Omega [(y-y_0)\,u - xv]\,dx\,dy$$
$$= \iint_\Omega (yu - xv)\,dx\,dy - y_0 LI, \tag{5.1}$$

where I is the horizontal momentum density:

$$I = \frac{1}{L}\iint_\Omega u\,dx\,dy. \tag{5.2}$$

From (5.1) and (5.2) we see that in general

$$A(y_0) = A(0) - y_0 I. \tag{5.3}$$

Now there will be one value of y_0 for which $A(y_0)$ vanishes, namely

$$y_a = A(0)/I. \tag{5.4}$$

Let us call this the *level of action* of the wave train. It is in fact the level at which we would have to add or subtract a small amount of linear momentum if we were to maintain the wave form constant, save only for a small increase or decrease in the wave amplitude.

A simple analogy may make this clearer. Consider a uniform circular disk of radius a rolling along a horizontal plane surface with angular velocity σ, as in figure 3. Taking the density per unit cross-sectional area as unity, the total mass of the disk is

$$M = \iint r\,dr\,d\theta = \pi a^2 \tag{5.5}$$

and the angular momentum of the disk about its centre is

$$A = \iint r^2\sigma\,.\,r\,dr\,d\theta = \tfrac{1}{2}\pi a^4\sigma. \tag{5.6}$$

The horizontal speed U of the centre of mass, however, is equal to $a\sigma$, so that its linear momentum is

$$I = Mu = \pi a^3\sigma. \tag{5.7}$$

Therefore the point P about which it has zero angular momentum is at a distance y_a above the centre, where

$$y_a = A/I = \tfrac{1}{2}a. \tag{5.8}$$

This also is the centre of impact, at which a horizontal impulsive force would have to be applied to start the disk rolling from rest, or to stop it dead, without slipping or bouncing.

If we imagine a thin slice of the disk, of uniform thickness, shaved off and concentrated to a point at P by purely internal forces, it could then be released and would fly

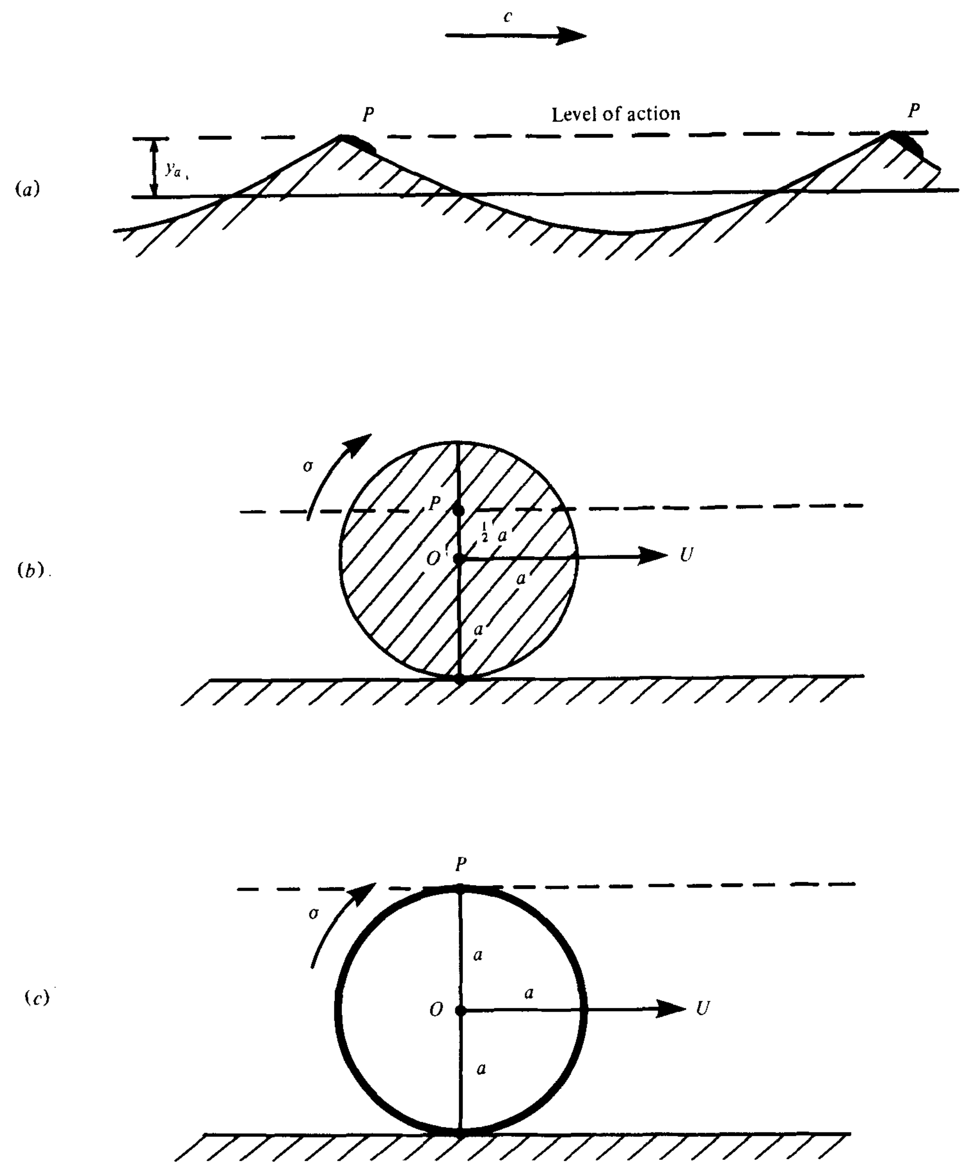

FIGURE 3. Comparison of (a) a steep progressive wave to (b) a rolling disk or (c) a bowling hoop.

off horizontally. The only effect would be to slow down the rolling motion of the disk by a small amount.

In the case of a circular hoop, the corresponding point P would be on the circumference, at the highest point.

Now in the case of a breaking wave, a mass of fluid is thrown forwards at a level near the crest. This is often observed to form a whitecap and to settle down into a quasi-steady state, with the wave progressing almost unchanged in form save presumably for a steady decrease in wave energy. Our previous calculations suggest that this would be impossible if the whitecap were not close to the level of action of the wave. This in turn implies that the height of the wave must be equal to, or slightly exceed, the level of action.

Now for low waves we have $A(0) = \frac{1}{4}a^2c$ while $I = E/c = \frac{1}{2}ga^2/c$, and so from (5.4)

$$y_a = c^2/2g. \tag{5.9}$$

Making use of the linear dispersion relation $c^2 = (g/k)\tanh kh$ we find

$$y_a = \frac{1}{2k}\tanh kh. \tag{5.10}$$

In deep water this reduces to

$$y_a = 1/2k = L/4\pi. \tag{5.11}$$

In other words the level of action is about one-twelfth of a wavelength above the mean surface level. Obviously it is impossible for low waves to attain this level, which may explain why such waves do not support whitecaps. The values of $y_{\max}$ and y_a for waves of finite amplitude will be calculated in § 8 and 9.

In shallow water (5.10) reduces to

$$y_a = \tfrac{1}{2}h \tag{5.12}$$

suggesting that steady whitecaps will not exist when the wave height is less than about one-half of the undisturbed depth h.

6. The Eulerian viewpoint: evaluation of $\bar{A}_E$

In this section we shall be concerned with the Eulerian angular momentum

$$LA_E = \iint_{\Omega_0} (yu - xv)\,dx\,dy, \tag{6.1}$$

where now Ω_0 is not a volume moving with the fluid but a fixed space or area independent of the time. In equations (2.3) to (2.5) we may conveniently choose $f(y) = x_0$, a constant, so that the lateral boundaries of the region are the vertical planes $x = x_0$ and $x = x_0 + L$. We may then consider the Eulerian-mean angular momentum

$$L\bar{A}_E = \overline{\int_{x_0}^{x_0+L}\int_{-h}^{y_s}(yu - xv)\,dx\,dy}, \tag{6.2}$$

where a horizontal bar denotes the time-average. In this expression we may take $x_0 = 0$ without loss of generality.

Note first that for a progressive wave

$$\overline{\int_0^L\int_{-h}^{y_s} xv\,dx\,dy} = 0 \tag{6.3}$$

precisely. Because v is a function of $(x-ct)$ and since the lateral boundaries are fixed, we can reverse the order of averaging with respect to x and t, that is, we may integrate first with respect to y and t keeping x fixed. But since the total vertical momentum vanishes it appears that for fixed x

$$\overline{\int_{-h}^{y_s} v\,dy} = 0, \tag{6.4}$$

from which (6.3) follows. In other words the contribution of the vertical velocity to the Eulerian-mean angular momentum is zero.

We have then to evaluate

$$L\bar{A}_E = \overline{\int_0^L \int_{-h}^{y_s} yu\, dx\, dy}. \tag{6.5}$$

Denoting by ϕ the velocity potential and transforming the integral by Green's theorem, we have

$$L\bar{A}_E = \iint \frac{\partial}{\partial x}(y\phi)\, dx\, dy = \int_B y\phi\, dy, \tag{6.6}$$

where the line-integral is taken anticlockwise round the boundary B. Now the integral along the bottom $y = -h$ vanishes, since $dy = 0$. The integrals along the two sides cancel by periodicity, and we are left with

$$L\bar{A}_E = -\int_{x=0}^{L} \phi\, d(\tfrac{1}{2}y^2) = \int_{x=0}^{L} \tfrac{1}{2}y_s^2\, d\phi \tag{6.7}$$

after integrating by parts, since $[\frac{1}{2}y_s\,\phi]_0^L$ vanishes by (2.2). Equation (6.7) shows that $\bar{A}_E$ must be at least third order in the wave steepness a.

To be more precise, it will be found convenient to introduce the velocity potential Φ and stream function Ψ of the motion relative to axes moving with the phase-speed c. Thus we write

$$\left.\begin{aligned} \Phi &= \phi - c(x-ct) \\ \text{and} \qquad \Psi &= \psi - c(y+h), \end{aligned}\right\} \tag{6.8}$$

where ϕ and ψ are the velocity and stream function in the original frame of reference. The arbitrary constants have been adjusted so that Ψ and ψ both vanish on the bottom $y = -h$. Also we choose Φ to vanish at a wave crest. In the new frame of reference the flow is independent of the time. Since

$$d\phi = d\Phi + c\, dx \tag{6.9}$$

and Φ runs from 0 to $-cL$ as x runs from 0 to L we obtain

$$L\bar{A}_E = \int_0^L \tfrac{1}{2}y_s^2\, c\, dx - \int_0^{cL} \tfrac{1}{2}y_s^2\, d\Phi. \tag{6.10}$$

The first integral is clearly related to the potential energy density

$$V = \frac{1}{L}\int_0^L \tfrac{1}{2}gy_s^2\, dx \tag{6.11}$$

so we can write

$$\bar{A}_E = \frac{Vc}{g} - \frac{1}{L}\int_0^{cL} \tfrac{1}{2}y_s^2\, d\Phi. \tag{6.12}$$

Now since the motion is periodic and irrotational, x and y may be expressed as Fourier series in Φ in the usual way:

$$\left.\begin{aligned} x - ct &= -\Phi/c + \sum_{n=1}^{\infty} a_n \sin(nk\Phi/c)\frac{\cosh(nk\Psi/c)}{\sinh(nk\Psi_s/c)}, \\ y + h &= -\Psi/c + \sum_{n=1}^{\infty} a_n \cos(nk\Phi/c)\frac{\sinh(nk\Psi/c)}{\sinh(nk\Psi_s/c)} \end{aligned}\right\} \tag{6.13}$$

where $k = 2\pi/L$ and Ψ_s is the (constant) value of Ψ at the free surface. We note that

$$\Psi_s = \int_{-h}^{y_s} \frac{\partial \Psi}{\partial y}\, dy = \int_{-h}^{y_s} (u-c)\, dy = I - ch, \tag{6.14}$$

where I is the horizontally averaged momentum or mass-flux, and h as before is the mean undisturbed depth. Hence

$$\Psi_s/c + h = I/c. \tag{6.15}$$

The free surface is therefore given by

$$y_s = -I/c + \sum_{n=1}^{\infty} a_n \cos(nk\Phi/c) \tag{6.16}$$

and the crest-to-trough wave height is $2a$, where

$$a = a_1 + a_3 + a_5 + \ldots. \tag{6.17}$$

Substituting for y_s in (6.12) we obtain finally

$$\bar{A}_E = \frac{Vc}{g} - \frac{I^2}{2c} - \tfrac{1}{4}c \sum_{n=1}^{\infty} a_n^2. \tag{6.18}$$

Consider for example the case of waves in deep water. In this case we have, to order a^2,

$$a_1 = a, \quad a_2 = a^2 k \tag{6.19}$$

and

$$\frac{Vc}{g} = \tfrac{1}{4}a^2 c, \quad I = \tfrac{1}{2}a^2 ck. \tag{6.20}$$

Hence

$$\bar{A}_E = 0 \tag{6.21}$$

to order a^2. Calculations carried to higher order show that in deep water

$$\bar{A}_E \doteqdot \tfrac{1}{4}a^4 k^2 c. \tag{6.22}$$

To reconcile the differing values of the Eulerian angular momentum $\bar{A}_E$ and the Lagrangian angular momentum $\bar{A}_L$ we note that, at the instants $\sigma t = 0, \pi$ when the boundaries Ω_0 and $\Omega(t)$ of the two masses of fluid coincide, the angular momenta contained in Ω_0 and Ω must obviously be the same. But when $\sigma t \neq 0, \pi$ the lateral boundaries of $\Omega(t)$ which are given by $k\alpha = 0, 2\pi$ in (3.1) do not concide with $x = 0, L$; in fact we have $x = f(y), f(y) + L$, where

$$f(y) = a\frac{\cosh k(y+h)}{\sinh kh} \sin \sigma t. \tag{6.23}$$

The resulting difference in the angular momenta is given by equation (2.8), with $f_1 = 0$ and $f_2 = f$. To order a^2 this is simply

$$\Delta A = \int_{-h}^{0} - v\, \delta x\, dy \tag{6.24}$$

where v is the vertical velocity:

$$v = -a\sigma \frac{\sinh k(y+h)}{\sinh kh} \sin \sigma t \tag{6.25}$$

and δx is the width of Ω_{12}, that is to say $f(y)$ approximately. All together we have

$$\Delta A = \int_{-h}^{0} \frac{a^2\sigma \sinh 2k(y+h)}{2\sinh^2 kh} \sin^2 \sigma t \, dy$$

$$= \tfrac{1}{2}a^2 c \sin^2 \sigma t. \tag{6.26}$$

On average, then,

$$\Delta \bar{A} = \tfrac{1}{4}a^2 c, \tag{6.27}$$

which agrees with the calculated difference between $\bar{A}_L$ and $\bar{A}_E$.

7. Angular momentum flux

The dynamical equation (4.10) was seen to apply to a fluid mass $\Omega(t)$ moving with the fluid. Consider now the fluid contained in the region Ω_0 fixed in space and momentarily coincident with $\Omega(t)$. From geometrical considerations it is clear that

$$\frac{D}{Dt}\iint_{\Omega(t)} (yu - xv)\, dx\, dy = \frac{\partial}{\partial t}\iint_{\Omega_0} (yu - xv)\, dx\, dy + \int_B (yu - xv)\, u_n\, ds, \tag{7.1}$$

where u_n denotes the outward normal component of the fluid velocity at the boundary B. The last term in (7.1) represents the flux of angular momentum across B. Since

$$u_n\, ds = u\, dy - v\, dx \tag{7.2}$$

it can also be written

$$LF_A = \int_B (yu - xv)\,(u\, dy - v\, dx). \tag{7.3}$$

If we are considering a mass of fluid contained in a region Ω^*, part of whose boundary is fixed and part moving with the fluid, as in the calculation of A_E, then the contour integral in (7.1) is to be taken only along the fixed portion B^* of B.

The flux across a vertical plane x = constant is given by

$$F = -\int_{-h}^{y_s} (yu - xv)\, u\, dy. \tag{7.4}$$

To evaluate the mean value of F during one wave period we may write $x = 0$ and find, to second order,

$$F = -\int_{-h}^{0} y\overline{u^2}\, dy = \bar{A}_L \cdot \tfrac{1}{2}c\left(1 + \frac{k^2h^2}{\sinh^2 kh}\right), \tag{7.5}$$

where $\bar{A}_L$ is given by (3.6). Both in deep water ($kh \gg 1$) and in shallow water ($kh \ll 1$), the factor multiplying A_L reduces to the linear group velocity

$$c_g = \tfrac{1}{2}c\left(1 + \frac{2kh}{\sinh 2kh}\right) \tag{7.6}$$

but not at intermediate depths.

The explanation lies in the fact that the injection of angular momentum into the waves depends not only on the flux across vertical planes but also on the moment of the gravitational forces in the interior and the moment of the pressure at the bottom. The combined contribution from these forces does not vanish in general, and particularly not near the front of an advancing wave train.

8. The long-time average of $A_L(t)$

In § 3 we found the time average of the Lagrangian angular momentum on the assumption that the fluid displacements were of order a, and before the drift velocity had carried the particles far from their original positions. We now investigate what happens over a long period, when the profile of a line of particles, originally vertical, has been radically distorted by the mass-transport velocity. In finding an answer we shall incidentally arrive at a satisfactory definition of the Lagrangian-mean angular momentum for waves of finite amplitude.

In figure 1, let $A_L(t)$ denote the angular momentum of the fluid mass which at $t = 0$ was contained between the two vertical planes $x = 0$ and L. As in § 2 we have

$$A_L(t) = A_L(0) + \Delta A, \tag{8.1}$$

where

$$\Delta A = \iint_{\bar{\Omega}_{12}} (-v)\, dx\, dy = \iint_{\Omega_{12}} v\, dx\, dy \tag{8.2}$$

and $\bar{\Omega}_{12}$ is the complementary volume to Ω_{12}; the integral of v over the whole volume $(\Omega_{12} + \bar{\Omega}_{12})$ must be zero. In (8.2) we now write

$$\left.\begin{aligned} v &= \frac{\partial \Phi}{\partial y} = q^2 \frac{\partial y}{\partial \Phi}, \\ dx\, dy &= q^{-2}\, d\Phi\, d\Psi, \end{aligned}\right\} \tag{8.3}$$

where q is the particle speed in the steady motion referred to axes moving with horizontal velocity c. This yields

$$\Delta A = \iint_{\bar{\Omega}_{12}} \frac{\partial y}{\partial \Phi}\, d\Phi\, d\Psi = \int (y_1 - y_0)\, d\Psi, \tag{8.4}$$

where y_0 and y_1 are the values of y on $x = 0$ and on the left-hand boundary of $\bar{\Omega}_{12}$.

We now propose to investigate the long-time average $\text{l.t.}\,\overline{\Delta A}$. Since the particles move along streamlines $\Psi = $ constant in the present frame of reference, the order of the time-averaging and of the integration with respect to Ψ can be reversed, and so from (8.4) we have

$$\text{l.t.}\,\overline{\Delta A} = \int \overline{(y - y_0)}\, d\Psi. \tag{8.5}$$

But if ds denotes any element of a streamline we have $q = ds/dt = d\Phi/ds$ and hence

$$dt = q^{-1}\, ds = q^{-2}\, d\Phi. \tag{8.6}$$

It follows that

$$\overline{y - y_0} = \frac{1}{\tau} \int_0^\tau (y - y_0)\, dt = \frac{1}{\tau} \int_0^\tau (y - y_0)\, q^{-2}\, d\Phi, \tag{8.7}$$

where τ is the orbital period, namely the time taken for a particle to travel one complete wavelength along a streamline:

$$\tau = \int_0^\tau dt = \int_0^{cL} q^{-2}\, d\Phi. \tag{8.8}$$

Note that τ is a function of Ψ. In fact $(c-L/\tau)$ is the local mass-transport velocity (Rayleigh 1876). From (8.7) and (8.8) we have

$$\overline{y-y_0} = \int (y-y_0)\, q^{-2}\, d\Phi \Big/ \int q^{-2}\, d\Phi. \tag{8.9}$$

In the first integral it is convenient to write $y-y_0 = (y-Y)-(y_0-Y)$, where Y is some function of Ψ at our disposal, and then (8.9) becomes

$$\overline{y-y_0} = \int (y-Y)\, q^{-2}\, d\Phi \Big/ \int q^{-2}\, d\Phi - (y_0 - Y). \tag{8.10}$$

This expression can now be substituted in (8.5). It must be emphasized, however, that the time-averaging process applied strictly in only two cases; first, over a small integral number of cycles, in low waves such that all terms smaller than $(ak)^2$ are negligible; secondly, over times such that typical particles have been carried many wavelengths from their original position. We see that in this case convergence towards the final value will be more rapid for waves of higher amplitude.

For simplicity consider waves in deep water, when

$$\left.\begin{aligned} x-ct &= -\Phi/c - \sum_{n=1}^{\infty} a_n \sin(n\Phi/c)\, e^{-n\Psi/c}, \\ y+H &= -\Psi/c + \sum_{n=1}^{\infty} a_n \cos(n\Phi/c)\, e^{-n\Psi/c}, \end{aligned}\right\} \tag{8.11}$$

the wavelength L being taken as 2π, so $k = 1$, and the free surface being given by $\Psi = 0$. To make the mean level zero we must take

$$0 = \frac{1}{2\pi}\int y\, dx = -H + \frac{1}{2}\sum_{n=1}^{\infty} n a_n^2. \tag{8.12}$$

Straightforward claculation now yields

$$\begin{aligned} q^{-2} &= \left(\frac{\partial y}{\partial \Phi}\right)^2 + \left(\frac{\partial y}{\partial \Psi}\right)^2 \\ &= \frac{1}{c^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b_m b_n \cos(m-n)\Phi/c\, e^{-(m+n)\Psi/c}, \end{aligned} \tag{8.13}$$

where

$$b_0 = 1, \quad b_n = n a_n, \quad n = 1, 2, 3, \ldots. \tag{8.14}$$

Hence

$$\frac{1}{L}\int_0^{cL} (y+H+\Psi/c)\, q^{-2}\, d\Phi = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} a_m b_n b_{m+n}\, e^{-2(m+n)\Psi/c} \tag{8.15}$$

and

$$\frac{1}{L}\int_0^{cL} q^{-2}\, d\Phi = \sum_{n=0}^{\infty} b_n^2\, e^{-2n\Psi/c}. \tag{8.16}$$

Denoting these two sums by Σ and Σ' respectively, we have from (8.10)

$$\begin{aligned} \overline{y-y_0} &= \Sigma/\Sigma' - (y+H+\Psi/c) \\ &= \Sigma/\Sigma' - \sum_{n=1}^{\infty} a_n\, e^{-n\Psi/c}. \end{aligned} \tag{8.17}$$

Hence

$$\text{l.t.}\,\overline{\Delta A} = \int_0^\infty (\Sigma/\Sigma')\, d\Psi - \sum_{n=1}^{\infty} a_n c/n. \tag{8.18}$$

Now from (8.1), on taking mean values, we have

$$\text{l.t.}\,\bar{A}_L = A_L(0) + \text{l.t.}\,\overline{\Delta A}. \tag{8.19}$$

The initial value $A_L(0)$ is equal to $A_E(0)$ and can be shown (see appendix A) to be given by

$$A(0) = \frac{Vc}{g} - \frac{I^2}{2c} - \tfrac{1}{2}c \sum_{n=1}^{\infty} a_n^2 + c \sum_{n=1}^{\infty} \frac{a_n}{n} \tag{8.20}$$

precisely. Substituting into (8.19), we obtain

$$\text{l.t.}\,\bar{A}_L = \frac{Vc}{g} - \frac{I^2}{2c} - \tfrac{1}{2}c \sum_{n=1}^{\infty} a_n^2 + Q, \tag{8.21}$$

where

$$Q = \int_0^\infty \Sigma/\Sigma'\, d\Psi. \tag{8.22}$$

To lowest order in the wave steepness we have

$$\left.\begin{aligned} b_0 &= 1, \\ b_1 &= a_1 = a, \\ b_2 &= 2a_2 = 2a^2, \end{aligned}\right\} \tag{8.23}$$

and so

$$\left.\begin{aligned} \Sigma &= a_1 b_0 b_1 e^{-2\Psi/c} = a^2 e^{-2\Psi/c}, \\ \Sigma' &= b_0^2 = 1, \end{aligned}\right\} \tag{8.24}$$

giving

$$Q = \int_0^\infty a^2 e^{-2\Psi/c} = \tfrac{1}{2}a^2 c. \tag{8.25}$$

Hence

$$\text{l.t.}\,\bar{A}_L = \tfrac{1}{2}a^2 c. \tag{8.26}$$

Remarkably, this is the same as (3.6) and (3.18), showing that, to order a^2, *the long-term average of the Lagrangian angular momentum is equal to the short-term average over one wave period*. This is because the period of the motion, though not exactly the same for all the different particles, is very nearly so when the wave steepness is small.

9. Numerical results

For general values of the wave steepness ak the Fourier coefficients a_n can be most easily calculated by the method described in Longuet-Higgins (1978), and the corresponding Lagrangian-mean angular momentum l.t. $\bar{A}_L$ can be found from (8.21). From this in turn we can determine the level of action y_a.

Table 1 shows first some numerical values of c, $y_{\max}$, I, T and V for representative values of the dimensionless wave amplitude a (k being taken as unity). These were

a	c	y_{max}	I	T	V
·00	1·0000	·0000	·00000	·00000	·00000
·05	1·0011	·0512	·00125	·00062	·00062
·10	1·0050	·1051	·00497	·00250	·00249
·15	1·0113	·1616	·01111	·00562	·00556
·20	1·0202	·2212	·01955	·00997	·00977
·25	1·0317	·2842	·03006	·01551	·01502
·30	1·0460	·3517	·04226	·02210	·02110
·35	1·0630	·4250	·05539	·02944	·02760
·40	1·0822	·5079	·06750	·03652	·03350
·41	1·0860	·5266	·06933	·03765	·03436
·42	1·0896	·5461	·07068	·03850	·03498
·43	1·0923	·5675	·07116	·03887	·03514
·44	1·0926	·590	·07026	·03838	·03464
·4431	1·0923	·5966	·0701	·0383	·0346

TABLE 1. Speed, crest-height, momentum and energy of deep-water waves.

calculated in the following sequence. First, $c = 1/C_0$, where C_0 is the lowest Fourier coefficient C_n, in the notation of Longuet-Higgins (1978). Then

$$b_n = C_n/C_0, \quad n = 0, 1, 2, 3, \ldots, \tag{9.1}$$

$$a_n = b_n/n, \quad n = 1, 2, 3, \ldots, \tag{9.2}$$

and

$$H = \frac{1}{2}\sum_{n=1}^{\infty} a_n b_n \tag{9.3}$$

from (8.12).

Now it is easy to show, since

$$LI = \int_0^L \int_{-h}^{y_s} u\,dy\,dx = \int_0^L \int_{-h}^{y_s} (u-c)\,dy\,dx + c\int_0^L \int_{-h}^{y} dy\,dx, \tag{9.4}$$

that for water of finite depth h

$$I = (\Psi_s - \Psi_h) + ch. \tag{9.5}$$

In deep water, when $\Psi_s = 0$ and (by (8.11))

$$\lim_{y\to-\infty} (y + \Psi/c) = -H, \tag{9.6}$$

we deduce that

$$I = cH, \tag{9.7}$$

which gives us the fourth column in table 1. Then setting $\Phi = 0$, $\Psi = 0$ in (8.11) we obtain

$$y_{\max} = -H + \sum_{n=1}^{\infty} a_n. \tag{9.8}$$

For the kinetic energy we have Levi-Cività's relation

$$T = \tfrac{1}{2}cI \tag{9.9}$$

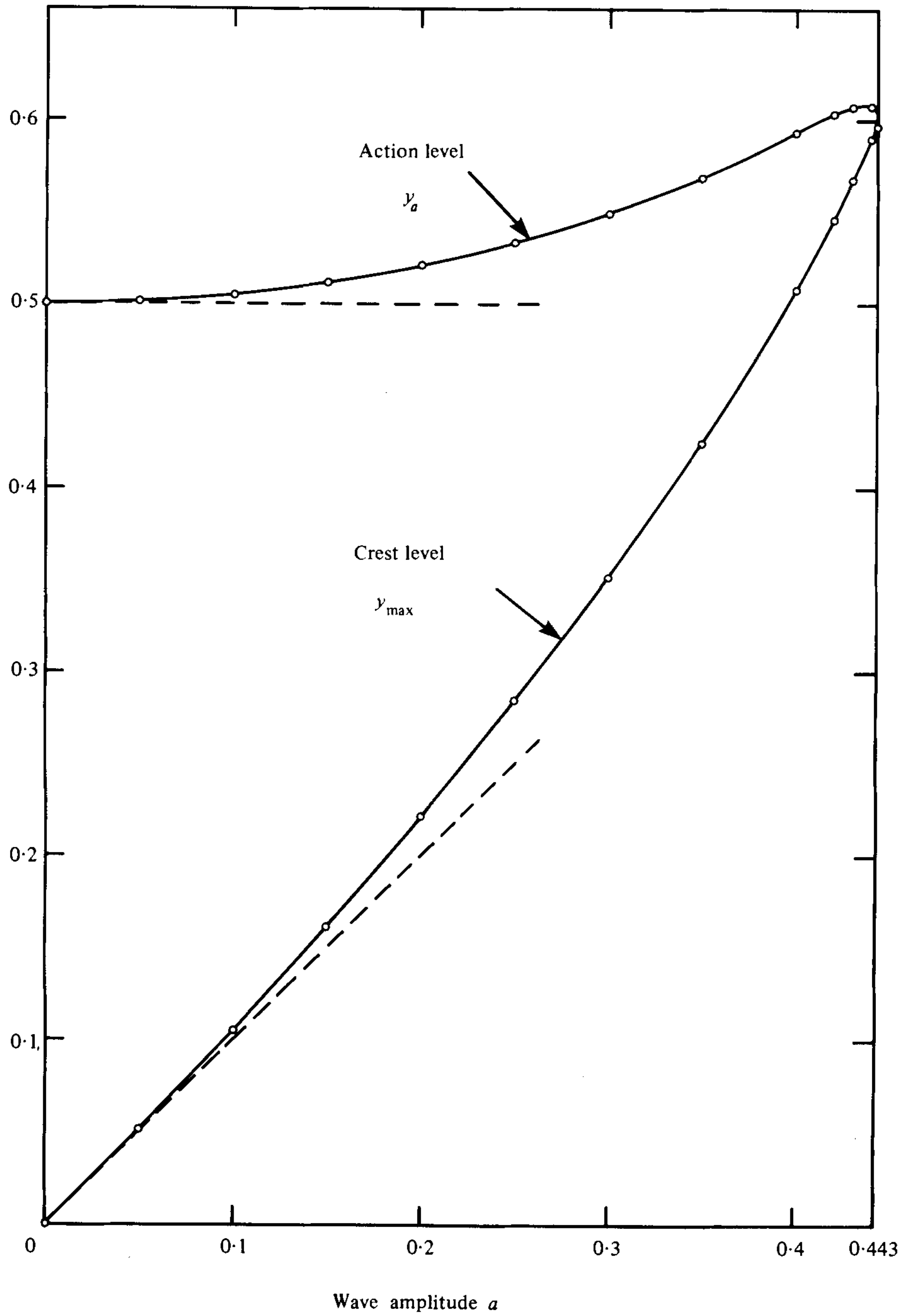

FIGURE 4. The level of action y_a and the crest height y_{max} shown as functions of the wave amplitude a (when $L = 2\pi$, $k = 1$) for waves in deep water.

a	S	$\bar{A}_E$	Q	$\bar{A}_L$	y_a
·00	·00000	·000000	·00000	·00000	·50000
·05	·00124	·000002	·00125	·00063	·50125
·10	·00492	·000025	·00495	·00251	·50504
·15	·01086	·000128	·01099	·00568	·51147
·20	·01875	·000408	·01919	·01018	·52070
·25	·02811	·001003	·02907	·01602	·53300
·30	·03823	·002099	·04020	·02319	·54875
·35	·04799	·003904	·05157	·03148	·56846
·40	·05522	·006541	·06106	·03999	·59248
·41	·05595	·007132	·06228	·04144	·59765
·42	·05595	·007694	·06487	·04260	·6027
·43	·05588	·008127	·06300	·04319	·6069
·44	·05482	·00810	·0619	·0427	·608
·4431	·0527	·0078	·0596	·0411	·586

TABLE 2. Angular momentum of deep-water waves.

(Levi-Cività 1924; equation (B) of Longuet-Higgins 1975), and lastly in table 1

$$V = -\tfrac{1}{2}H^2 + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} a_m a_{m+n} b_n + \frac{1}{2}\sum_{n=2}^{\infty}\sum_{m=0}^{n-1} a_m a_{n-m} t_n. \tag{9.10}$$

In table 2 we show the corresponding values of S, $\bar{A}_E$, Q, l.t.$\bar{A}_L$ and y_a, where

$$S = \tfrac{1}{2}c\sum_{1}^{\infty} a_n^2, \tag{9.11}$$

$$\bar{A}_E = (V - \tfrac{1}{2}H^2)c - \tfrac{1}{2}S \tag{9.12}$$

and Q is given by (8.22), where Σ and Σ' are the sums on the right-hand sides of (8.15) and (8.16). Finally we have

$$\text{l.t.}\bar{A}_L = (V - \tfrac{1}{2}H^2)c + Q - S \tag{9.13}$$

and by definition

$$y_a = \text{l.t.}\bar{A}_L/I. \tag{9.14}$$

In figure 4 the values of $y_{\max}$ and y_a are shown as functions of the dimensionless wave amplitude a. At small wave amplitudes we have of course $y_{\max} \sim a$, but for larger amplitudes the increasing sharpness of the wave crests as compared with the wave troughs necessitates that $y_{\max}$ must increase more rapidly than a. For the steepest waves, when $a = 0{\cdot}443$ (Longuet-Higgins 1975), the value of $y_{\max}$ can be very simply determined through the relation

$$y_{\max} = \tfrac{1}{2}c^2. \tag{9.15}$$

Taking $c^2 = 1{\cdot}1931$ we find $y_{\max} = 0{\cdot}596$.

On the other hand the calculated value of y_a, which for infinitesimal wave amplitudes equals 0·5, rises gradually until near the end of the interval of a, when (like the phase speed c) it has a maximum followed by a slight downturn. Its final value appears to be about 0·59, very close to the final value of $y_{\max}$.

10. Discussion

The first question raised by the preceding calculation is whether for the limiting wave in deep water $y_{\max}$ and y_a should be precisely equal. For this conjecture there is as yet no theoretical support. Admittedly, in a limiting wave, the contribution of the fluid near the wave crest to the two integrals in (8.9) becomes relatively important, since the fluid lingers in that area. More precisely, in the well-known Stokes corner-flow we have Φ and Ψ proportional to $r^{\frac{3}{2}}$, while q is proportional to $r^{\frac{1}{2}}$. Hence in (8.9) the contribution to the numerator is $O(r^{\frac{5}{2}})$ and to the denominator $0(r^{\frac{3}{2}})$. The quotient is therefore $O(r)$ and on integrating with respect to Ψ in (8.17) we have a contribution $O(r^{\frac{5}{2}})$ which is finite. Therefore the contribution to the angular momentum from the neighbourhood of the crest, though possibly important, is not necessarily dominant.

The fact that in steep waves $y_a \doteqdot y_{\max}$ has possible implications both for the generation and the decay of wind-waves. As pointed out in § 5 it appears possible, in such steep waves, to increase or decrease the linear momentum by the application of a horizontal force *at the surface* without appreciably changing the level of action of the wave train. Hence there would be less tendency (than in other situations where a concentrated force is applied) for the energy to be scattered into different wavelengths. In other words we suggest that the neighbourhood of a sharp wave crest is precisely the place to apply a localized horizontal stress, if momentum or energy are to be added smoothly; and, because of the pressure difference induced by air-flow separation at a sharp crest, this is precisely the place where wave-generating forces are indeed likely to be applied.

Similarly in the case of wave decay. In a whitecap, the flow is evidently strongly sheared, and the density of the whitecap may be less than that of unaerated water, as is found in hydraulic jumps. Hence the whitecap may be considered as a superposed mass 'surf-riding' on the fluid below, and replaceable in some respects by a normal pressure distribution. The fact that under some conditions a progressive wave can support a 'spilling' whitecap while progressing smoothly in a quasi-steady state may be attributed to the fact that the whitecap is situated not far from the level of action of the wave.

Of course this discussion implies that the resulting applied force acts *horizontally*. In fact, if the whitecap remains in contact with the wave, the reaction between it and the wave at subsequent times will have a vertical component.† However the downwards vertical component of the applied force must be balanced by an increased pressure at greater depths, say a wavelength or more below the surface, and this increased pressure will be more or less uniformly distributed with regard to horizontal distance. Then it can be seen that, provided that the vertical component of the surface force is symmetrically distributed with regard to the wave crest, it will have zero moment about a point below the crest. Hence the total couple exerted by the vertical component of the additional forces will vanish.

Gerstner waves. When the previous arguments are applied to a Gerstner wave, in which the flow is no longer irrotational but has a strong negative vorticity, the results are somewhat different. In such a theoretical wave (see Lamb 1932, § 251) the particles

† In the analogy with the rolling disk, it is as though the mass thrown off remained in contact with the disk, and were supported by it at a point immediately above the disk's centre. There is consequent loss of energy to the whole system, though momentum is conserved.

move in closed circular orbits and their mean forwards drift vanishes precisely. The contribution to the Lagrangian angular momentum from the particle orbits is therefore positive (and equal to $\frac{1}{2}a^2c$, approximately) while the total linear momentum I vanishes. The level of y_a is therefore infinite. Although the Gerstner wave has a limiting form (with a *cusp* at the crest) nevertheless it is evidently impossible for the wave surface ever to reach the level of action, which is infinitely high. We may infer, first, that it must be extremely difficult to generate a steady Gerstner wave by pressure forces applied at the surface. Secondly, a Gerstner wave, even when breaking, will not support a whitecap.

These remarks serve to emphasize still further the unnatural character of a Gerstner wave, in which the surface shear is strongly negative. The arguments may be generalized to waves on a shearing current of any form. Then it is easy to see at least qualitatively that waves on a shearing stream with a *positive* vorticity must have a *lower* level of action than those on otherwise still water. Hence they will tend to break at a lower wave amplitude. The same conclusion has been reached on other grounds by Banner & Philips (1974).

Spin and action density. Naeser (1978) has pointed out that the orbital 'spin' or angular momentum of particles in a wave train has the same dimensions as wave *action*. This can most easily be seen from the definition of action as the line integral of the momentum (Lamb 1929, § 104). The dimensions of wave action are therefore mass × velocity × length, similar to angular momentum. We have also for low waves

$$\bar{A}_L = \tfrac{1}{4}a^2c = Ec/2g, \tag{10.1}$$

where E is the energy density $\frac{1}{2}ga^2$. But in deep-water waves, for example, $\sigma^2 \doteqdot gk$ or $\sigma c \doteqdot g$; hence

$$2\bar{A}_L = E/\sigma, \tag{10.2}$$

the well-known expression for the action density (Hasselmann 1963). It follows that, since in wave–wave interactions and also in wave–current interactions the total wave action is conserved, the same must be true of the Lagrangian angular momentum density also.

However it is important that the two quantities (action and angular momentum) should not be confused. In the case of irrotational waves, for example, the orbital angular momentum involves multiplying the momentum by a distance *normal* to the path of the particle, and not along it, while the contribution from the Stokes drift, which is of the same order of magnitude, again involves a different length.

In this paper, detailed calculations have been given so far only for surface waves in deep water. Similar arguments and methods of calculation will also apply, with suitable modification, to solitary waves and to internal waves in an unbounded, stratified fluid. However for *periodic* waves in water of finite depth the argument is less attractive since the appropriate frame of reference appears not to be uniquely determined.

Appendix A. The precise evaluation of $A(0)$

The general expression (2.5) for the angular momentum may be written

$$LA = \iint_\Omega \left(y\frac{\partial\phi}{\partial x} - x\frac{\partial\phi}{\partial y}\right) dx\,dy \tag{A 1}$$

and by a transformation similar to (4.6), but with ϕ in place of P, this becomes

$$LA = -\int_B \tfrac{1}{2}(x^2+y^2)\,d\phi \tag{A 2}$$

where B is the boundary of Ω. That is to say

$$LA = -\int_B \tfrac{1}{2}(x^2+y^2)\,d\Phi - \int_B \tfrac{1}{2}(x^2+y^2)\,c\,dx, \tag{A 3}$$

Φ being the velocity potential of the steady flow as seen in a frame moving with the phase-speed c. But, since x is single-valued,

$$\int_B \tfrac{1}{2}x^2\,dx = 0. \tag{A 4}$$

the contribution from the term y^2 along the bottom is

$$-\int \tfrac{1}{2}h^2\,d\phi = -\tfrac{1}{2}h^2\,[\phi]_{x=0}^{L} = 0 \tag{A 5}$$

by equation (2.2). Since the motion is periodic in x, the two lateral boundaries $x = f(y)$ and $x = f(y)+L$ contribute

$$-\int_f \tfrac{1}{2}[(x+L)^2-x^2]\,d\phi = -L\int_f x\,d\phi. \tag{A 6}$$

So we are left with

$$LA = \int \tfrac{1}{2}x_b^2\,d\phi - \int \tfrac{1}{2}(x_s^2+y_s^2)\,d\Phi + \tfrac{1}{2}\int y_s^2 c\,dx - L\int x\,d\phi. \tag{A 7}$$

Let us take the case when the lateral boundaries correspond to Φ = constant, say $\Phi = \Phi_0$ and $\Phi_0 + cL$. Then in the first two integrals the limits of integration are the same. Moreover

$$\int_f x\,d\phi = \int_f xc\,dx = \tfrac{1}{2}c(x_s^2-x_b^2). \tag{A 8}$$

All together then we have

$$A = \frac{Vc}{g} - \frac{1}{2L}\int_0^{cL} \{(x_s^2-x_b^2)+y_s^2\}\,d\Phi - \tfrac{1}{2}c(x_s^2-x_b^2). \tag{A 9}$$

Now, substituting for x and y the Fourier series (6.11) and recalling (6.13), we obtain

$$A = \frac{Vc}{g} - \frac{I^2}{2c} - \tfrac{1}{2}c\sum_{n=1}^{\infty} a_n^2 + R + S, \tag{A 10}$$

where

$$\begin{aligned} R &= \frac{1}{L}\int_{\Phi_0}^{\Phi_0+cL} (\Phi/c)\sum_{n=1}^{\infty} a_n \sin(nk\Phi/c)\,\frac{\cosh(nk\Psi_s/c)-1}{\sinh(nk\Psi_s/c)} \\ &= -\frac{c}{k}\sum_{n=1}^{\infty}\frac{a_n}{n}\cos(nk\Phi_0/c)\tanh(nk\Psi_s/2c) \end{aligned} \tag{A 11}$$

and

$$S = (c^2t-\Phi_0)\sum_{n=1}^{\infty} a_n \sin(nk\Phi_0/c)\tanh(nk\Psi_s/2c) + \tfrac{1}{2}c\sum_{n=1}^{\infty} a_n^2\sin^2(nk\Phi_0/c). \tag{A 12}$$

The crest-to-crest angular momentum corresponds to $\Phi_0 = 0$, giving $S = 0$ identically. So we have

$$A(0) = \frac{Vc}{g} - \frac{I^2}{2c} - \tfrac{1}{2}c \sum_{n=1}^{\infty} a_n^2 - \frac{c}{k} \sum_{n=1}^{\infty} \frac{a_n}{n} \tanh(nk\Psi_s/c) \tag{A 13}$$

and in particular for deep water when $k\Psi_s/2c = -\infty$ we have

$$A(0) = \frac{Vc}{g} - \frac{I^2}{2c} - \tfrac{1}{2}c \sum_{n=1}^{\infty} a_n^2 + \frac{c}{k} \sum_{n=1}^{\infty} \frac{a_n}{n} \tag{A 14}$$

as was to be proved.

Appendix B. Historical note

The angular momentum of water waves seems to have been first adduced by V. V. Shuleikin (1954, § 4) in connection with the generation of waves by wind. However, it is clear that he did not include in his estimates the angular momentum of the Stokes drift.

Wave-spin has been considered from a very general point of view as an antisymmetric tensor by Jones (1973, equation 28*b*). What we have called the 'orbital angular momentum' he has called the 'wave-spin density'. He assumes however that a particle has no average linear momentum, so that the contribution from the mass-transport (that is, the moment of the mean linear momentum) would be negligible. We have seen that this is not justified.†

The author's brother (H. C. Longuet-Higgins, personal communication) drew his attention to the fact that the contribution of wave crests and troughs to the Eulerian angular momentum tended to cancel out. This led the author to the more accurate evaluation of this quantity given in § 5. Later H. Naeser (1978) suggested the equivalence of wave spin and wave action, an idea which is discussed in § 10.

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† It should be noted that on pp. 744 and 747 of Jones's paper 'Buchwalder (1972)' should be Buchwald (1972); on p. 741 'Eckert' should be Eckart; on pp. 738 and 747 the date of Longuet-Higgins and Stewart (1969) is actually 1964. A more appropriate reference would be to an earlier paper by the same authors (*J. Fluid Mech.* vol. 8 (1960), p. 565).

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